independent to the same extent as the spirals used. The phase difference remains exactly constant at all frequencies where their axial ratios are equal to one.

III. Theory of Operation

The fact that the phase of a circularly polarized antenna is dependent upon its rotation has been successfully used by Brown and Dodson in developing a novel antenna design. This rotational phase interdependence is the underlying principle in the network under discussion.

Fig. 2 is a general representation of the network. Energy enters the tee junction from port 1 and splits between arms 2 and 3. P_1 and P_2 are polarizers (such as arithmetic spirals) which convert the input to the circularly polarized mode. The polarizers transfer the energy through equal lengths of transmission line to ports 2 and 3.

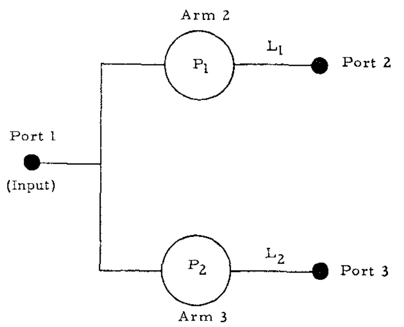

Fig. 2—A three-port network with constant phase difference properties.

Let us consider the ideal case where both polarizers are identical and always provide axial ratios equal to one. Assume that the polarizers initially have the same rotation with respect to an arbitrary fixed axis; *i.e.*, have the same orientation in space. Since arms 2 and 3 are identical in every way, there will be no phase difference at their outputs. However, if one of the polarizers is rotated with respect to the other, a phase difference will appear at ports 2 and 3. This phase difference is independent of frequency and is numerically equal to the angle of rotation. Symbolically, if the polarizers were originally at an angle α_1 with respect to the x axis and one of them, say P_2, were rotated to an angle α_2, then the phase difference between ports 2 and 3 is

$$\phi_{23} = \alpha_2 - \alpha_1. \tag{1}$$

Eq. (1) may be proved as follows: Consider the x and y components of a circularly polarized wave E_1,

$$E_{1x} = E \cos \omega \left(t - \frac{z}{v}\right)$$

$$E_{1y} = E \cos \left[\omega \left(t - \frac{z}{v}\right) - \frac{\pi}{2}\right]$$

where

z = the direction of propagation
v = the velocity of the wave.

In the plane $z=0$ these equations reduce to

$$E_{1x} = E \cos \omega t$$

$$E_{1y} = E \cos \left(\omega t - \frac{\pi}{2}\right) = E \sin \omega t.$$

At any time t_1 the angle between the resultant electric vector and the x axis is

$$\alpha_1 = \tan^{-1} \frac{E_{1y}}{E_{1x}} = \tan^{-1} \frac{E \sin \omega t_1}{E \cos \omega t_1}$$

$$\alpha_1 = \omega t_1. \tag{2}$$

Now let us consider the x and y components of a second wave E_2. E_2 is identical to E_1 in every way except that it is shifted in phase by a constant amount, ϕ. At $z=0$,

$$E_{2x} = E \cos (\omega t + \phi)$$

$$E_{2y} = E \cos \left(\omega t - \frac{\pi}{2} + \phi\right) = E \sin (\pi t + \phi).$$

The angle of the electric vector at the same time t_1 is, in this case,

$$\alpha_2 = \tan^{-1} \frac{E_{2y}}{E_{2x}} = \tan^{-1} \frac{E \sin (\omega t_1 + \phi)}{E \cos (\omega t_1 + \phi)}$$

$$\alpha_2 = \omega t_1 + \phi. \tag{3}$$

The fact that $\alpha_2 \neq \alpha_1$ means that one wave is rotated with respect to the other; *i.e.*, one polarizer is rotated with respect to the other. Taking the angular difference,

$$\alpha_2 - \alpha_1 = \omega t_1 + \phi - \omega t_1$$

$$\alpha_2 - \alpha_1 = \phi,$$

which proves (1).

IV. The Network in Practice

In practice, (1) is true only for perfect circular polarization. However, slight errors due to small ellipticities should not destroy its usefulness.

The fact that the outputs of arms 2 and 3 are circularly polarized may be less convenient than if they were coaxial lines. It would be useful then to terminate arms 2 and 3 with depolarizers which reconvert the circularly polarized mode. If these depolarizers are identical and have the same rotation, none of the properties discussed will be affected.

In any configuration such as the above at least one serious problem is to be expected, that of transducing from the polarizing elements to the intervening transmission line. If spiral elements are used, the problems of higher-order mode excitation and proximity of the transmission line walls to the spiral conductors will have to be solved before any useful component can be built.

V. Effect of Unequal Power Split

The amplitudes at ports 2 and 3 are completely independent of the phase relationship. This is borne out by (2) and (3) which show that the angle the electric vector makes with the x axis is a function of frequency, time, and initial phase only.

VI. Phase Difference Variation

Fig. 3 is a drawing of what a practical network might look like. The coaxial tee feeds two spirals backed by conical cavities. They, in turn, direct their energy down cylindrical waveguides which are terminated in two additional spirals. Rotary joints before and after one of the polarizers allow rotation. Finally, the output appears at a pair of coaxial connectors. The numerical value of the phase difference may be simply controlled by rotating the polarizer sandwiched between the rotary joints to the desired angle.

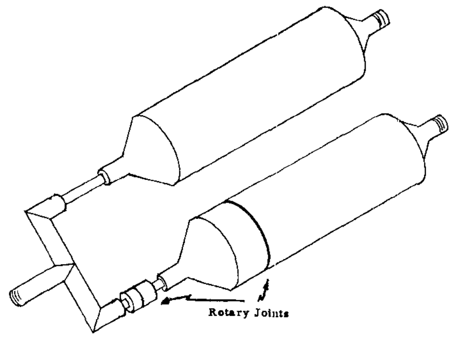

Fig. 3—Example of a practical network.

VII. Conclusion

A network has been described which is theoretically frequency-independent. In practice, however, several limitations crop up. Among these are the bandwidth of the spiral elements, the bandwidth of the waveguide used, and spiral ellipticity.

The most serious design problem to be expected is the transformation from the spiral elements to the circular waveguide.

It is hoped that a network will be simulated in the near future which will adequately test these limitations.

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A Simple Formula for Calculating Approximate Values of the First Zeros of a Combination Bessel Function Equation*

The solution of many problems in microwave theory, particularly those relating to waveguides having curved boundary surfaces, is dependent upon a determination of the zeros of the Bessel function equation

$$J_p(x)N_p(kx) - J_p(kx)N_p(x) = 0 \tag{1}$$

where J_p and N_p are respectively the Bessel functions of the first and second kinds, of order p. In a majority of the cases arising in waveguide theory, the parameters k and p

* Received September 28, 1962.

are real and positive, but may be fractional or integral.

While tables of the zeros of (1) do exist for certain ranges of p and k (see [1], [2], [3]) these are in general rather limited in extent, particularly for large and fractional values of p and k. Hence, for the solution of problems involving non-tabulated values of the parameters it is necessary either to interpolate existing tabulations, or to solve (1) numerically. In many such instances much of the labor may be avoided by the use of a simple relation, derived by the author in the course of developing an approximate theory of propagation in rectangular waveguide wound into a helical form [for the exact treatment of this problem see [4], which also contains extensive tables of the zeros of (1)]. On the basis of the assumptions

a) that only the TE_{10} mode is propagated,
b) that electrical lengths may be measured along the axis of the waveguide, and
c) that the pitch of the helix is negligible (see [4] for justification of this),

the following formula for the roots of (1) is derived:

$$x_0 \doteq \sqrt{\frac{\pi^2}{(k-1)^2} + \frac{4p^2}{(k+1)^2}} \cdot \qquad (2)$$

The values x_0 obtained from (2) are very close approximations to the first zeros of (1). The closeness of the approximation depends upon the particular values of k and p under consideration, and for a given case may be estimated by reference to the accompanying Fig. 1. If the point determined by (k, p) lies within the central cross-hatched region, the resultant value of x_0 will in general be within ± 1 per cent of the exact value, though if it lies within the region bounded by the dashed curve the lower limit may drop to -1.5 per cent. If the point lies anywhere within the diagonally-hatched region the value of x_0 calculated from (2) will be within ± 5 per cent of the exact value.

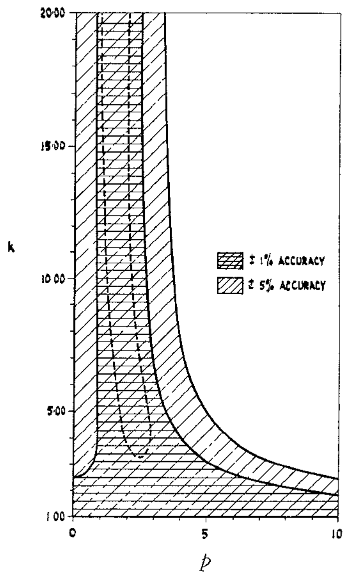

Fig. 1.

For the design engineer, accuracies within ± 1 per cent will often be adequate, and in such cases the use of (2) obviates the need for interpolation of tables or other tedious calculation. In other cases, where high accuracy is required, the use of (2) will quicly provide an excellent "first guess" which will permit a rapidly convergent numerical solution of (1). It may further be noted that, given any two of the three parameters p, k, x_0, the third may readily be calculated from (2) and the accuracy of the result determined from Fig. 1.

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Maximum Efficiency of a Two Arm Waveguide Junction*

It is well-known that the efficiency of a two-arm waveguide junction (2-port) depends upon the reflection coefficient Γ_L of the load with which one of the arms is terminated. The efficiency is known to vary between the limits 0 and η_m (maximum efficiency) as Γ_L assumes all possible values within the unit circle. However, there seems to be no published analysis from which one can determine the particular Γ_L giving maximum efficiency if the characteristics of the waveguide junction are known.

* Received October 11, 1962.

It can be shown[1] that the reflection coefficient Γ_M to give maximum efficiency can be calculated from

$$\Gamma_M = S_{22}^* + \frac{(1-S_{22}\Gamma_M)^* S_{11}S_{12}^* S_{21} + |S_{12}S_{21}|^2\Gamma_M}{(1-|S_{11}|^2)(1-S_{22}\Gamma_M) - S_{11}^* S_{12}S_{21}\Gamma_M}, \qquad (1)$$

where the asterisk * denotes the complex conjugate, the S-terms denote the scattering coefficients of the waveguide junction, and the load of reflection coefficient Γ_M terminates arm 2. The solution of (1) for Γ_M may be written

$$\Gamma_M = \frac{B}{2A}\left[1 \pm \sqrt{1 - \left(\frac{2|A|}{B}\right)^2}\right].$$

Where

$$A = S_{22} + S_{11}^*(S_{12}S_{21} - S_{11}S_{22}), \qquad (2)$$

and

$$B = 1 - |S_{11}|^2 + |S_{22}|^2 - |S_{12}S_{21} - S_{11}S_{22}|^2$$

in some cases, it is necessary to choose the algebraic sign in (2) to yield a value of Γ_M within the unit circle.

One Γ_M has been determined, the maximum efficiency η_M can be determined from the equation

$$\eta_M = \frac{Z_{01}}{Z_{02}} \cdot \frac{|S_{21}|^2(1-|\Gamma_M|^2)}{|1-S_{22}\Gamma_M|^2 - |(S_{12}S_{21}-S_{11}S_{22})\Gamma_M + S_{11}|^2}, \qquad (3)$$

where Z_{01} and Z_{02} are the characteristic impedances of arms 1 and 2, respectively, of the 2-arm waveguide junction.

It can be further shown[2] that the quantity A_I, the intrinsic attenuation (equivalent to the intrinsic insertion loss of Tomiyasu[3]) is given by

$$A_I = 10 \log_{10} \frac{1}{\eta_M} \cdot \qquad (4)$$

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[1] A convenient way to show this, is to postulate lossless tuners attached to both arms of the waveguide junction and adjusted for maximum power to the load. Under this condition, a conjugate match is obtained at each terminal surface in each waveguide lead. For simplicity, one may assume a non-reflecting generator and load without significant loss in generality. A straightforward analysis then leads to the stated result.

[2] This is shown in a paper entitled "Intrinsic Attenuation" which is in preparation by the correspondent.

[3] Kiyo Tomiyasu, "Intrinsic Insertion Loss of a Mismatched Microwave Network," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp 40–44; January 1955.